Stop talking and write down the Hamiltonian!

- I. B. Khriplovich, Novosibirsk State University (Russia), during a seminar at Berkeley.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1

(a)

First we determine how many visible photons are emitted from the surface of the sun-like star. The radiated power per unit area per unit wavelength $\frac{dR}{d\lambda}$ is given by the Planck distribution:

$$\frac{dR}{d\lambda} = \frac{2\pi hc^2}{\lambda^5(e^{hc/\lambda kT} - 1)}. \tag{1}$$

We want to convert this quantity into the number of photons per second per unit area per unit wavelength $\frac{dn}{d\lambda}$, which can be done by dividing $\frac{dR}{d\lambda}$ by the energy per photon:

$$E_{\text{photon}} = \hbar\omega = \frac{hc}{\lambda}. \tag{2}$$

We integrate over the visible spectrum to get photons per second per unit area, using $T = 5800\ K$ for the temperature of the sun-like star:

$$n = \int_{400\text{ nm}}^{700\text{ nm}} d\lambda \frac{2\pi c}{\lambda^4(e^{hc/\lambda kT} - 1)} = 6.3 \times 10^{25} \text{ photons s}^{-1}\text{ m}^{-2}. \tag{3}$$

The total number of photons given off by the sun per second N is n times the surface area of the star, which is roughly $4\pi R_S^2 = 6 \times 10^{18}\ m^2$. This gives us 3.8×10^{44} photons per second from the star! We scale this by the solid angle subtended by the observer's eye,

$$\frac{\pi r_{\text{eye}}^2}{4\pi R_{SE}^2},$$

and solve for R_{SE} such that the eye receives at least 250 photons. This gives us

$$R_{SE}^{\max} \approx 10^{19} \text{ m} \approx 1300 \text{ light years}.$$

(b)

The cosmic background radiation fills the universe roughly isotropically (there is no solid angle suppression), and the temperature of the radiation is $T = 2.74$ K. The incident number of photons per square cm on Penzias and Wilson's antenna is given by an equation similar to Eq. (3):

$$n = \int_0^\infty d\lambda \frac{2\pi c}{\lambda^4(e^{hc/\lambda kT} - 1)} = 2.6 \times 10^{12} \text{ photons s}^{-1}\text{ cm}^{-2}. \tag{4}$$

Problem 2

(a)

The maximum energy of a electron ejected by the photoelectric effect is given by:

$$\frac{hc}{\lambda} - \Phi, \tag{5}$$

where Φ is the work function. We have two data points to use in this relation, with which we can determine h:

$$\frac{hc}{0.2\mu m} - \Phi = 2.3 \text{ eV}$$

$$\frac{hc}{0.313\mu m} - \Phi = 0.9 \text{ eV}$$

Subtracting these equations and solving for h gives us

$$h \approx 2.6 \times 10^{-21} \text{ MeV} \cdot \text{s}.$$

Substituting this value of h back into one of the photoelectric effect equations gives us the work function

$$\Phi = 1.58 \text{ eV}.$$

(b)

Quantum efficiency of the photocathode is just the ratio of emitted electrons to incident photons. The number of photons is the power of light over the average energy per photon at this wavelength:

$$N_\gamma = \frac{P\lambda}{hc} = 1.6 \times 10^{15} \text{ photons/sec}.$$

The number of electrons is just the current over the charge per electron:

$$N_e = \frac{I}{q_e} = 6.2 \times 10^{12} \text{ electrons/sec.}$$

Taking the ratio gives us the quantum efficiency:

$$\text{QE} = 0.0039 = 0.39\%.$$

This was using the correct value for h. If you used the value of h you obtained from part (a) of this problem, you would find

$$\text{QE} = 0.0025 = 0.25\%.$$

Problem 3

(a)

For thermal radiation, the average energy per photon is given by:

$$\langle E \rangle = \frac{1}{n}\int_0^\infty dE \cdot E\frac{dn}{dE} \tag{6}$$

The energy per unit volume u is given by:

$$u = n\langle E \rangle = \int_0^\infty dE \cdot E\frac{dn}{dE} \tag{7}$$

We can employ the fact that from Eq. (7), $\frac{du}{dE} = E\frac{dn}{dE}$, which gives us

$$\frac{du}{dE}\frac{dE}{d\lambda} = E\frac{dn}{dE}\frac{dE}{d\lambda} \tag{8}$$

This reduces to our desired result, simply that:

$$\frac{du}{d\lambda} = E\frac{dn}{d\lambda} = \frac{hc}{\lambda}\frac{dn}{d\lambda}. \tag{9}$$

(b)

The total photon density n is given by:

$$n = \int_0^\infty d\lambda\frac{dn}{d\lambda} = \int_0^\infty d\lambda\frac{\lambda}{hc}\frac{du}{d\lambda}, \tag{10}$$

which gives us:

$$n = \int_0^\infty d\lambda\left(\frac{\lambda}{hc}\right)\frac{8\pi hc}{\lambda^5(e^{hc/\lambda kT}-1)} = 8\pi\int_0^\infty \frac{d\lambda}{\lambda^4(e^{hc/\lambda kT}-1)}. \tag{11}$$

Making the appropriate substitution $x \equiv hc/\lambda kT$ and $dx = -\frac{hc}{kT}\frac{d\lambda}{\lambda^2}$, we get the desired result:

$$n = 8\pi\left(\frac{kT}{hc}\right)^3\int_0^\infty dx\frac{x^2}{e^x-1} \approx (3.17\times10^{19}\ \text{eV}^{-3}\cdot\text{m}^{-3})(\text{kT})^3. \tag{12}$$

(c)

The average energy per photon is u/n. $u = 4R/c$ can be found from the Stefan-Boltzmann law (the factor of 4 comes from averaging over all angles, Rohlf Eqs. (3.17) and (3.18)),

$$u = 4R/c = \frac{4\sigma'(kT)^4}{c}. \tag{13}$$

Therefore, the average energy per photon is found from the ratio of Eqs. (13) and (12) to be:

$$\langle E \rangle = \frac{u}{n} = \frac{4\sigma'(kT)}{c\cdot(3.17\times10^{19}\ \text{eV}^{-3}\cdot\text{m}^{-3})}. \tag{14}$$

Plugging in the constants gives us:

$$\langle E \rangle \approx 2.7\ kT. \tag{15}$$

(d)

For the number density of photons, we obtain from Eq. (12):

$$n = (3.17\times10^{19}\ \text{eV}^{-3}\cdot\text{m}^{-3})[(8.62\times10^{-5}\ \text{eV/K})(2.74\ \text{K})]^3 \approx 4\times10^8\ \text{m}^{-3}.$$

For the energy density, we from Eq. (15) we find that

$$\langle E \rangle \approx 0.64\text{meV}.$$

Problem 4

(a)

Acceleration a for a circular orbit is given by:

$$a = \frac{v^2}{r}. \tag{16}$$

Angular momentum L, applying the Bohr quantization condition, is given by:

$$L = mvr = n\hbar. \tag{17}$$

Solving for v from Eq. (17) and substituting into Eq. (16), we obtain:

$$a = \frac{L^2}{m^2r^3} = \frac{n^2\hbar^2}{m^2r^3}. \tag{18}$$

The Bohr radius r is given by:

$$r = (4\pi\epsilon_0)\frac{n^2\hbar^2}{me^2}. \tag{19}$$

Using Eq. (19) in conjunction with Eq. (18), we find an expression for a in terms of fundamental constants:

$$a = \frac{n^2\hbar^2}{m^2r^3} = \left(\frac{1}{4\pi\epsilon_0}\right)^3\frac{me^6}{n^4\hbar^4}. \tag{20}$$

Using Eq. (20) in the classical expression for radiated power gives us:

$$P = \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}a^2 = \left(\frac{1}{4\pi\epsilon_0}\right)^7\frac{2}{3}\frac{m^2e^{14}}{c^3n^8\hbar^8}. \tag{21}$$

(b)

The energy E_n in the n^{th} level of the Bohr atom is given by:

$$E_n = -\frac{\alpha^2mc^2}{2n^2} = -\frac{-13.6\text{ eV}}{n^2}, \tag{22}$$

so the energy radiated in a $n \to n-1$ transition is:

$$\Delta E = -13.6\text{ eV}\left(\frac{1}{n^2} - \frac{1}{(n-1)^2}\right). \tag{23}$$

The decay rate γ is then the ratio of the radiated power (we assume that the electron is in the n^{th} orbit until the moment it decays) to the energy difference between the levels:

$$\gamma = \frac{P}{\Delta E}. \tag{24}$$

If we plug in our result from part (a) and use $n = 2$, we get

$$\gamma = 10^8\text{ s}^{-1}.$$

If the decay is from the n^{th} level to the $(n-m)^{\text{th}}$ level, we merely adjust the energy difference in Eq. (23):

$$\Delta E = -13.6\text{ eV}\left(\frac{1}{n^2} - \frac{1}{(n-m)^2}\right). \tag{25}$$

and employ this equation in Eq. (24). Qualitatively, we see that if the energy difference is greater and the power radiated is the same, the decay rate will decrease. This is an example of the limitations of the Bohr model, since although it correctly predicts the order of magnitude of the transition rates it does not correctly predict the dependence of transition rates on the energy difference between levels, which actually scales as ω^3.

(c)

There is an energy time uncertainty principle, which can be derived from $\Delta x\Delta p \geq \hbar/2$ in the following hand-waving fashion:

$$\Delta E\Delta t = \left(\frac{p}{m}\Delta p\right)\left(\frac{m}{p}\Delta x\right) = \Delta x\Delta p.$$

We use the lifetime $(1/\gamma)$ as the uncertainty in time, and then find for the uncertainty in energy:

$$\Delta E = \frac{\gamma\hbar}{2} \tag{26}$$

The value of ΔE is $(10^8\text{ s}^{-1})\cdot(197.3\text{ MeV fm}\cdot(3\times10^{23}\text{fm/s})^{-1})$, or 6×10^{-8} eV. The energy difference between the first and second levels in hydrogen is 10 eV, so the linewidth is smaller than the energy difference by nine orders of magnitude!

Problem 5

In calculations involving the Bohr model, the electron mass is replaced by the reduced mass of the muon-proton system, which is near the mass of the muon:

$$m = \frac{m_\mu m_p}{m_\mu + m_p} = 95 \text{ MeV}.$$

Energy levels in the Bohr model are linear with respect to the electron mass, and are given by

$$E_n = -\frac{13.6 \text{ eV}}{n^2} \cdot \frac{95 \text{ MeV}}{0.5 \text{ MeV}} \tag{27}$$

for the muon-proton system.

(a)

A free muon decays with a characteristic lifetime 2.2×10^{-6} s, primarily in the mode:

$$\mu \to e^- + \bar{\nu}_e + \nu_\mu.$$

If the capture probability was large, it would make the lifetime shorter compared to the lifetime of a muon (at rest), since the two rates would add.

(b)

If the capture probability was small enough to be neglected, then the effect of time dilation would lengthen the lifetime of the muon (since in the muon-proton system, the muon has a characteristic velocity αc), compared to a muon at rest.

Problem 6

If $\mathcal{O}$ is divided by the characteristic impedance of free space Z_0, we get $(2\alpha)^{-1}$. So

$$\mathcal{O} = \frac{Z_0}{2\alpha},$$

probably.

Problem 7

(a)

The typical electron velocity in the Bohr model is $v = \alpha c$, for the deuteron we replace α by α_s. So we have $v = \alpha_s c = 3 \times 10^7$ m/s for both the proton and neutron.

(b)

The reduced mass in the deuteron is roughly $m_p/2$, so the nuclear "Bohr radius" r is given by:

$$r = \frac{2\hbar c}{m_p c^2 \alpha_s} \approx \frac{2 \cdot 197.3 \text{ MeV} \cdot \text{fm}}{0.1 \cdot 938 \text{ MeV}} = 4 \text{ fm}.$$

(c)

The binding energy of the deuteron is roughly $\frac{1}{2}\alpha_s^2 mc^2 = 2$ MeV.

Problem 8

First, we equate the relativistic centripetal force to the electrostatic force acting on the electron:

$$\frac{ke^2}{r^2} = \frac{\gamma m v^2}{r}, \tag{28}$$

and proceed to solve for the radius:

$$r = \frac{ke^2}{\gamma m v^2}. \tag{29}$$

We can also use the Bohr quantization condition

$$pr = n\hbar$$

in conjunction with the relativistic expression for the momentum

$$p = \gamma m v$$

to find the radius:

$$r = \frac{\hbar}{\gamma m v}. \tag{30}$$

Setting Eqs. (29) and (30) equal, we can solve for the velocity, and we obtain the desired result $v = \alpha c$.